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1992 J. Phys. A: Math. Gen. 25 L101

(http://iopscience.iop.org/0305-4470/25/3/004)

J. Phys. A: Math. Gen. 25 (1992) L101–L108. Printed in the UK

LETTER TO THE EDITOR

Are q-bosons suitable for the description of correlated fermion pairs?

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Received 22 July 1991, in final form 25 November 1991

Abstract. In a single-j shell we consider fermion pair and multipole operators coupled to zero angular momentum. The commutation relations of these operators can be satisfied up to first-order corrections by suitably defined q-bosons, onto which the fermion pair operators are mapped. After performing the same mapping to a simple pairing Hamiltonian, we prove that the pairing energies are also correctly reproduced up to the same order. The small parameter used ($T = \ln Q$) is found to be inversely proportional to the size of the shell.

Quantum algebras [1–5], which from the mathematical point of view are Hopf algebras [6], are currently attracting much attention in physics, especially after the introduction of the q-deformed harmonic oscillator [7, 8] and its equivalent forms [9]. Initially used for solving the quantum Yang–Baxter equation [10], they have already been used in conformal field theories [11, 12], in the description of squeezed states [13–15] and spin chains [16, 17], as well as in describing rotational spectra of deformed nuclei [18, 19], superdeformed nuclei [20], and diatomic molecules [21], as well as in describing vibrational spectra of diatomic molecules [22, 23]. In this framework q-bosons can be introduced in different equivalent ways [7–9, 24]. They satisfy commutation relations which differ from the standard boson commutation relations, to which they reduce in the limit $q \to 1$.

On the other hand, it is well known in nuclear physics that correlated fermion pairs in a single-j shell [25–29] or several non-degenerate j-shells [30–32] satisfy commutation relations which resemble boson commutation relations including corrections due to the presence of the Pauli principle. This fact has been the cause for the development of boson mapping techniques (see the recent reviews of [33, 34] and references therein), by which the description of systems of fermions in terms of bosons is achieved. In recent years boson mappings have attracted additional attention in nuclear physics as a necessary tool in providing a theoretical justification for the success of the phenomenological interacting boson model [35–37] and its various extensions (see [38–40] for recent overviews), in which low-lying collective states of medium and heavy mass nuclei are described in terms of bosons.

From the above observations it is clear that both q-bosons and correlated fermion pairs satisfy commutation relations which resemble the standard boson commutation

0305-4470/92/030101+08\$04.50

relations but they deviate from them, due to the q-deformation in the former case and to the Pauli principle in the latter. A question is thus created: are q-bosons suitable for the approximate description of correlated fermion pairs? In particular, is it possible to construct a boson mapping in which correlated fermion pairs are mapped onto q-bosons, in a way that the q-boson operators approximately satisfy the same commutation relations as the correlated fermion pair operators? In this letter we show for the simple case of SU(2) that such a mapping is possible.

Let us consider the single-j shell model [25–29]. One can define fermion pair and multipole operators as

$$A^{+}_{JM} = \frac{1}{\sqrt{2}} \sum_{mm'} (jmjm'|JM) a^{+}_{jm} a^{+}_{jm'} \tag{1}$$

$$B_{JM} = \frac{1}{\sqrt{2J+1}} \sum_{mm'} (jmj-m'|JM)(-1)^{j-m'} a^{+}_{jm} a_{jm'} \tag{2}$$

with the following definitions

$$A_{JM} = [A^{+}_{JM}]^{+} \qquad B^{+}_{JM} = [B_{JM}]^{+}. \tag{3}$$

In the above a^{+}_{jm} (a_{jm}) are fermion creation (annihilation) operators and $(jmjm'|JM)$ are the usual Clebsch–Gordan coefficients.

The pair and multipole operators given above satisfy commutation relations which correspond to the $\mathrm{SO}(2(2j+1))$ algebra [25–29]. In the present letter, however, we will restrict ourselves to fermion pairs coupled to angular momentum zero. The relevant commutation relations take the form

$$[A_0, A^{+}_0] = 1 - \frac{N_{\mathrm{F}}}{\Omega} \tag{4}$$

$$\left[\frac{N_{\mathrm{F}}}{2}, A^{+}_0\right] = A^{+}_0 \tag{5}$$

$$\left[\frac{N_{\mathrm{F}}}{2}, A_0\right] = -A_0 \tag{6}$$

where N_{F} is the number of fermions, $2\Omega = 2j+1$ is the size of the shell, and

$$B_0 = N_{\mathrm{F}}/\sqrt{2\Omega}. \tag{7}$$

With the identifications

$$J_{+} = \sqrt{\Omega} A^{+}_0 \qquad J_{-} = \sqrt{\Omega} A_0 \qquad J_0 = \frac{N_{\mathrm{F}} - \Omega}{2} \tag{8}$$

equations (4)–(6) take the form of the usual SU(2) commutation relations

$$[J_{+}, J_{-}] = 2J_0 \qquad [J_0, J_{+}] = J_{+} \qquad [J_0, J_{-}] = -J_{-}. \tag{9}$$

An exact boson mapping of the SU(2) algebra is given by [29, 33]

$$A^{+}_0 = a^{+}_0 \sqrt{1 - \frac{n_0}{\Omega}} \qquad A_0 = \sqrt{1 - \frac{n_0}{\Omega}}\, a_0 \qquad N_{\mathrm{F}} = 2n_0 \tag{10}$$

where a_0^+ (a_0) are boson creation (annihilation) operators carrying angular momentum zero and n_0 is the number of these bosons.

The simplest pairing Hamiltonian one can consider has the form [25–28]

$$H = -G\Omega A_0^+ A_0. \tag{11}$$

The Casimir operator of SU(2) can be written as [25–28]

$$\{A_0^+, A_0\} + \frac{\Omega}{2}\left(1 - \frac{N_{\rm F}}{\Omega}\right)^2 = \frac{\Omega}{2} + 1 \tag{12}$$

while the pairing energy takes the form [25–28]

$$\frac{E}{(-G\Omega)} = \frac{N_{\rm F}}{2} - \frac{N_{\rm F}^2}{4\Omega} + \frac{N_{\rm F}}{2\Omega}. \tag{13}$$

Our aim is to check if there is a boson mapping for the operators A_0^+, A_0 and $N_{\rm F}$ in terms of q-deformed bosons, having the following properties:

(i) The mapping is simpler than that of equation (10), i.e. to each fermion pair operator A_0^+, A_0 corresponds a bare q-boson operator and not a boson operator accompanied by a square root (the Pauli reduction factor).

(ii) The commutation relations (4)–(6) are satisfied up to a certain order.

(ii) The pairing energies of equation (13) are reproduced up to the same order.

In recent work q-numbers are defined as

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} \tag{14}$$

where q can be real ($q = {\rm e}^\tau$, where τ real) or a phase ($q = {\rm e}^{{\rm i}\tau}$, with τ real). The q-deformed harmonic oscillator [7, 8] is defined in terms of the creation and annihilation operators a^+ and a and the number operator N, which satisfy the commutation relations

$$[N, a^+] = a^+ \qquad [N, a] = -a \qquad aa^+ - q^{\mp 1}a^+a = q^{\pm N}. \tag{15}$$

An immediate consequence of (15) is that

$$a^+a = [N] \qquad aa^+ = [N+1]. \tag{16}$$

The Hamiltonian of the q-deformed harmonic oscillator is

$$H = \frac{\hbar\omega}{2}(aa^+ + a^+a) \tag{17}$$

and its eigenvalues are

$$E(n) = \frac{\hbar\omega}{2}([n] + [n+1]). \tag{18}$$

For q being a phase, the commutator of a and a^+ takes the form

$$[a, a^+] = [N+1] - [N] = \frac{\cos[(2N+1)\tau/2]}{\cos(\tau/2)}. \tag{19}$$

In physical situations τ is expected to be small (i.e. of the order of 0.01), as in the cases of [18–23]. Therefore in equation (19) one can take Taylor expansions of the functions appearing there and thus find an expansion of the form

$$[a, a^+] = 1 - \frac{\tau^2}{2}(N^2 + N) + \frac{\tau^4}{24}(N^4 + 2N^3 - N) - \cdots. \tag{20}$$

We remark that the first-order corrections contain not only a term proportional to N, but in addition a term proportional to N^2, which is larger than N. Thus one cannot make the simple mapping

$$A_0 \to a \qquad A_0^+ \to a^+ \qquad N_{\mathrm{F}} \to 2N \tag{21}$$

because then one cannot get the commutation relation (4) correctly up to the first order of the corrections. The same problem appears in the case in which q is real as well. In addition, by making the simple mapping of equation (21) the pairing Hamiltonian can be written as

$$\frac{H}{-G\Omega} = a^+ a = [N]. \tag{22}$$

In the case of small τ, one can again take Taylor expansions of the trigonometric (hyperbolic) functions appearing in the definition of the q-numbers for q being a phase (real) and thus obtain the following expansion

$$\begin{aligned}[N] = N &\pm \frac{\tau^2}{6}(N - N^3) + \frac{\tau^4}{360}(7N - 10N^3 + 3N^5) \\ &\pm \frac{\tau^6}{15120}(31N - 49N^3 + 21N^5 - 3N^7) + \cdots\end{aligned} \tag{23}$$

where the upper (lower) sign corresponds to q being a phase (real). We remark that while the first-order corrections in equation (13) are proportional to N_{F}^2 and N_{F}, here the first-order corrections are proportional to N and N^3. Thus neither of the pairing energies can be reproduced correctly by this mapping.

However, a different version of the q-harmonic oscillator can be obtained by defining [8, 9] the operators b, b^+ through the equations

$$a = q^{1/2} b q^{-N/2} \qquad a^+ = q^{1/2} q^{-N/2} b^+. \tag{24}$$

Equation (15) then gives

$$[N, b^+] = b^+ \qquad [N, b] = -b \qquad bb^+ - q^2 b^+ b = 1. \tag{25}$$

By using the symbol $Q = q^2$ and introducing the Q-number

$$[x]_Q = \frac{Q^x - 1}{Q - 1} \tag{26}$$

we find the analogue of equation (16)

$$b^+b = [N]_Q \qquad bb^+ = [N+1]_Q. \tag{27}$$

The Hamiltonian of the corresponding deformed harmonic oscillator has the form

$$H = \frac{\hbar\omega}{2}(bb^+ + b^+b) \tag{28}$$

the eigenvalues of which are

$$E(n) = \frac{\hbar\omega}{2}([n]_Q + [n+1]_Q). \tag{29}$$

It should be noticed at this point that the definition of q-number given in equation (26) was historically the first to be introduced in the framework of q-analysis [41], while the last of the commutation relations of equation (25) has been studied by Kuryshkin [24]. For convenience from now on we will call the numbers of equation (26) the Q-numbers, while the numbers of equation (14) we will call the q-numbers.

From the above relations, it is clear that the following commutation relation holds

$$[b, b^+] = [N+1]_Q - [N]_Q = Q^N. \tag{30}$$

Defining $Q = \mathrm{e}^T$ this can be written as

$$[b, b^+] = 1 + TN + \frac{T^2N^2}{2} + \frac{T^3N^3}{6} + \cdots. \tag{31}$$

We remark that the first-order correction is proportional to N. Thus, by making the boson mapping

$$A_0^+ \to b^+ \qquad A_0 \to b \qquad N_{\mathrm{F}} \to 2N \tag{32}$$

one can satisfy equation (4) up to the first order of the corrections by determining $T = -2/\Omega$.

We should now check if the pairing energies (equation (13)) can be found correctly up to the same order of approximation when this mapping is employed. The pairing Hamiltonian in this case takes the form

$$\frac{H}{-G\Omega} = b^+b = [N]_Q. \tag{33}$$

Defining $Q = \mathrm{e}^T$ it is instructive to construct the expansion of the Q-number of equation (26) in powers of T. Assuming that T is small and taking Taylor expansions in equation (26) one finally has

$$[N]_Q = N + \frac{T}{2}(N^2 - N) + \frac{T^2}{12}(2N^3 - 3N^2 + 1) + \frac{T^3}{24}(N^4 - 2N^3 + N^2) + \cdots. \tag{34}$$

Using the value of the deformation parameter $T = -2/\Omega$, determined above from the requirement that the commutation relations are satisfied up to first-order corrections, the pairing energies become

$$\frac{E}{-G\Omega} = N - \frac{N^2 - N}{\Omega} + \frac{2N^3 - 3N^2 + 1}{3\Omega^2} - \frac{N^4 - 2N^3 + N^2}{3\Omega^3} + \cdots. \tag{35}$$

The first two terms in the right-hand side of equation (35), which correspond to the leading term plus the first-order corrections, are exactly equal to the pairing energies of equation (13), since $N_F \to 2N$. We therefore conclude that through the boson mapping of equation (32) one can both satisfy the fermion pair commutation relations (4)–(6) and reproduce the pairing energies of equation (13) up to the first-order corrections.

The following comments are also in order.

(i) By studying the spectra of the two versions of the q-deformed harmonic oscillator, given in equations (18) and (29), one can easily draw the following conclusions: when compared to the usual oscillator spectrum, which is equidistant, the spectrum of the q-oscillator shrinks when q is a phase, while the spectrum of the Q-oscillator shrinks when $T < 0$. In a similar way, the spectrum of the q-oscillator expands for q real, while the spectrum of the Q-oscillator expands for $T > 0$. In physical situations [18–23] it has been found that the physically interesting results are obtained with q being a phase. Thus in the case of the Q-oscillator it is when $T < 0$ which is the physically interesting case. As we have already seen, it is exactly for $T = -2/\Omega < 0$ that the present mapping gives the fermion pair results.

Table 1. Pairing energies for shells of different size $2\Omega = 2j + 1$. For each Ω the classical (CL) results obtained from equation (13) (with $N_F = 2N$) and the Q-boson (Q) results obtained from equation (33) are shown. In the latter case the values of the deformation parameter $T = -2/\Omega$ are also shown.

Ω	11	11	16	16	22	22
T		0.182		0.125		0.091
N	CL	Q	CL	Q	CL	Q
1	1.000	1.000	1.000	1.000	1.000	1.000
2	1.818	1.834	1.875	1.883	1.909	1.913
3	2.455	2.529	2.625	2.661	2.727	2.747
4	2.909	3.109	3.250	3.349	3.455	3.508
5	3.182	3.592	3.750	3.955	4.091	4.203
6	3.273	3.995	4.125	4.490	4.636	4.838
7			4.375	4.963	5.091	5.418
8			4.500	5.380	5.455	5.947
9					5.727	6.430
10					5.909	6.871
11					6.000	7.274

(ii) It should be recalled that the pairing model under discussion is studied under the assumptions [25–28] that the degeneracy of the shell is large ($\Omega \gg 1$), that the number of particles is large ($N \gg 1$), and that one stays away from the centre of the shell ($\Omega - N = O(N)$). In order to check the accuracy of the present mapping in reproducing the pairing energies, we report in table 1 the results of some calculations for $\Omega = 11$ (the size of the nuclear fp major shell), $\Omega = 16$ (the size of the nuclear sdg major shell) and $\Omega = 22$ (the size of the nuclear pfh major shell), while in table 2 we report the results for the case $\Omega = 50$ (as an example of a large shell). In all cases good agreement between the classical pairing model results and the Q-Hamiltonian of equation (33) is obtained up to the point at which about one-quarter of the shell is filled. The deviations observed near the middle of the shell are expected, since there the expansion used breaks down.

Table 2. Same as table 1, for $\Omega = 50$, $T = -0.04$.

N	CL	Q	N	CL	Q	N	CL	Q
1	1.000	1.000	10	8.200	8.408	19	12.160	13.576
2	1.960	1.961	11	8.800	9.078	20	12.400	14.044
3	2.880	2.884	12	9.360	9.722	21	12.600	14.493
4	3.760	3.771	13	9.880	10.341	22	12.760	14.925
5	4.600	4.623	14	10.360	10.936	23	12.880	15.340
6	5.400	5.442	15	10.800	11.507	24	12.960	15.738
7	6.160	6.228	16	11.200	12.056	25	13.000	16.121
8	6.880	6.984	17	11.560	12.583			
9	7.560	7.710	18	11.880	13.090			

In conclusion, we have shown that an approximate mapping of the fermion pairs coupled to angular momentum zero in a single-j shell onto suitably defined q-bosons (called Q-bosons in this letter) is possible. The SU(2) commutation relations are satisfied up to the first-order corrections, while at the same time the eigenvalues of a simple pairing Hamiltonian are correctly reproduced up to the same order. The small parameter of the expansion, which is T (where $Q = e^T$), turns out to be negative and inversely proportional to the size of the shell.

The present results are an indication that suitably defined q-bosons could be used for describing systems of correlated fermions under certain conditions in a simplified way. The extension of the present method to fermion pairs coupled to non-zero angular momentum, which would allow for a fuller treatment of the single-j shell model, is under investigation. The construction of q-bosons which would *exactly* satisfy the fermion pair SU(2) commutation relations is also under investigation, using the powerful method of [42].

Support from the Greek Ministry of Research and Technology is gratefully acknowledged.

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